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Evaluating Fractional Fourier Series Expansions of Two Types of Matrix Fractional Functions

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***Abstract*: In this paper, based on a new multiplication of fractional analytic functions, we find the fractional Fourier series expansions of two types of matrix fractional functions. Matrix fractional Euler's formula and matrix fractional DeMoivre's formula play important roles in this article. In fact, our results are generalizations of ordinary calculus results.**

***Keyword*: New multiplication, fractional analytic functions, fractional Fourier series expansions, matrix fractional functions, matrix fractional Euler's formula, matrix fractional DeMoivre's formula.**

I. INTRODUCTION

Fractional calculus is a mathematical analysis tool used to study arbitrary order derivatives and integrals. It unifies and extends the concepts of integer order derivatives and integrals. Generally, many scientists do not know these fractional integrals and derivatives, and they have not been used in pure mathematical context until recent years. However, in the past few decades, the fractional integrals and derivatives have frequently appeared in many scientific fields such as mechanics, viscoelasticity, physics, economics and engineering [1-8].

Until now, the definition of fractional derivative is not unique. The commonly used definitions are Riemann-Liouvellie (R-L) fractional derivative, Caputo definition of fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, conformable fractional derivative, and Jumarie's modified R-L fractional derivative [9-13]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, we find the fractional Fourier series expansions of the following two types of matrix fractional functions:

$$Ln_\alpha(1+2rcos_\alpha(Ax^\alpha)+r^2) ,$$

$$arctan_\alpha\left(\frac{rsin_\alpha(Ax^\alpha)}{1+rcos_\alpha(Ax^\alpha)}\right),$$

where $0<\alpha\leq 1$, r is a real number, $|r|<1$, and A is a real matrix. In fact, our results are generalizations of classical calculus results.

II. PRELIMINARIES

At first, fractional analytic function is introduced.

Definition 2.1 ([14])**:** If x, x_0, and a_n are real numbers for all n, $x_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval

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containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.2 ([15])**:** If $0 < \alpha \le 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha}, \tag{1}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha}. \tag{2}$$

Then

$$f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha)$$
$$= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha}$$
$$= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m \right) (x - x_0)^{n\alpha}. \tag{3}$$

Equivalently,

$$f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha)$$
$$= \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n}$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m \right) \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n}. \tag{4}$$

Definition 2.3 ([16])**:** Let $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha p} = f_\alpha(x^\alpha) \otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the pth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha -1}$.

Definition 2.4 ([17])**:** Let $0 < \alpha \le 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} x^\alpha. \tag{5}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.5 ([18])**:** If $0 < \alpha \le 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha n}. \tag{6}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha 2n}, \tag{7}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha (2n+1)}. \tag{8}$$

Definition 2.6: If $0 < \alpha \le 1$, and A is a real matrix. The matrix α-fractional exponential function is defined by

$$E_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(A \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha n}. \tag{9}$$

And the matrix α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{10}$$

and

$$sin_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{11}$$

Theorem 2.7 (matrix fractional Euler's formula) ([19])**:** *If* $0 < \alpha \le 1$, $i = \sqrt{-1}$, *and* A *is a real matrix, then*

$$E_\alpha(iAx^\alpha) = cos_\alpha(Ax^\alpha) + i sin_\alpha(Ax^\alpha). \tag{12}$$

Theorem 2.8 (matrix fractional DeMoivre's formula)([20])**:** *If* $0 < \alpha \le 1$, p *is an integer, and* A *is a real matrix, then*

$$[cos_\alpha(Ax^\alpha) + i sin_\alpha(Ax^\alpha)]^{\otimes_\alpha p} = cos_\alpha(pAx^\alpha) + i sin_\alpha(pAx^\alpha). \tag{13}$$

III. MAIN RESULTS

In this section, we find the fractional Fourier series expansions of two types of matrix fractional functions. At first, we need a lemma.

Lemma 3.1: *If* $0 < \alpha \le 1$, r *is a real number, and* A *is a real matrix, then*

$$Ln_\alpha\left(1 + rE_\alpha(iAx^\alpha)\right) = Ln_\alpha([1 + 2r cos_\alpha(Ax^\alpha) + r^2]) + i \cdot arctan_\alpha\left(\frac{r sin_\alpha(Ax^\alpha)}{1 + r cos_\alpha(Ax^\alpha)}\right). \tag{14}$$

Proof $Ln_\alpha\left(1 + rE_\alpha(iAx^\alpha)\right)$

$= Ln_\alpha\left(1 + r cos_\alpha(Ax^\alpha) + i r sin_\alpha(Ax^\alpha)\right)$ (by matrix fractional Euler's formula)

$$= Ln_\alpha\left(\begin{array}{c}[1 + 2r cos_\alpha(Ax^\alpha) + r^2] \\ \otimes_\alpha \left[\left(1 + r cos_\alpha(Ax^\alpha)\right)\otimes_\alpha [1 + 2r cos_\alpha(Ax^\alpha) + r^2]^{\otimes_\alpha -1} + i r sin_\alpha(Ax^\alpha)\otimes_\alpha [1 + 2r cos_\alpha(Ax^\alpha) + r^2]^{\otimes_\alpha -1}\right]\end{array}\right)$$

$$= Ln_\alpha\left([1 + 2r cos_\alpha(Ax^\alpha) + r^2]\otimes_\alpha E_\alpha\left(i \cdot arctan_\alpha\left(\frac{r sin_\alpha(Ax^\alpha)}{1 + r cos_\alpha(Ax^\alpha)}\right)\right)\right)$$

$= Ln_\alpha([1 + 2r cos_\alpha(Ax^\alpha) + r^2]) + i \cdot arctan_\alpha\left(\frac{r sin_\alpha(Ax^\alpha)}{1 + r cos_\alpha(Ax^\alpha)}\right)$. q.e.d.

Theorem 3.2: *If* $0 < \alpha \le 1$, r *is a real number,* $|r| < 1$, *and* A *is a real matrix, then*

$$Ln_\alpha(1 + 2r cos_\alpha(Ax^\alpha) + r^2) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} r^{n+1} cos_\alpha((n+1)Ax^\alpha), \tag{15}$$

and

$$arctan_\alpha\left(\frac{r sin_\alpha(Ax^\alpha)}{1 + r cos_\alpha(Ax^\alpha)}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} r^{n+1} sin_\alpha((n+1)Ax^\alpha). \tag{16}$$

Proof Since $|r| < 1$, it follows that

$Ln_\alpha\left(1 + rE_\alpha(iAx^\alpha)\right)$

$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1}[rE_\alpha(iAx^\alpha)]^{\otimes_\alpha (n+1)}$

$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} r^{n+1} E_\alpha(i(n+1)Ax^\alpha)$ (by matrix fractional DeMoivre's formula)

$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} r^{n+1} cos_\alpha((n+1)Ax^\alpha) + i\sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} r^{n+1} sin_\alpha((n+1)Ax^\alpha). \tag{17}$$

Therefore, by Lemma 3.1

$$Ln_{\alpha}(1+2rcos_{\alpha}(Ax^{\alpha})+r^{2})=\sum_{n=0}^{\infty}(-1)^{n}\frac{1}{n+1}r^{n+1}cos_{\alpha}((n+1)Ax^{\alpha}),$$

and

$$arctan_{\alpha}\left(\frac{rsin_{\alpha}(Ax^{\alpha})}{1+rcos_{\alpha}(Ax^{\alpha})}\right)=\sum_{n=0}^{\infty}(-1)^{n}\frac{1}{n+1}r^{n+1}sin_{\alpha}((n+1)Ax^{\alpha})\ .$$ q.e.d.

IV. CONCLUSION

In this paper, we find the fractional Fourier series expansions of two types of matrix fractional functions. Matrix fractional Euler's formula and matrix fractional DeMoivre's formula play important roles in this article. In fact, our results are generalizations of classical calculus results. In the future, we will continue to use our methods to study the problems in applied mathematics and fractional differential equations.

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